التمرين 1 :

أحسب التكامل التالي : $A=\int_0^{\frac{\pi}{4}}\sqrt{\tan x}\,dx$.

الجواب

لدينا : $A=\int_0^{\frac{\pi}{4}}\sqrt{\tan x}\,dx$

$$=\int_0^{\frac{\pi}{4}}\left(1+\tan^2(x)\right)\sqrt{\tan x}\,dx-\int_0^{\frac{\pi}{4}}\tan^2(x)\sqrt{\tan x}\,dx$$

$$=\int_0^{\frac{\pi}{4}}\tan'(x)\sqrt{\tan x}\,dx-\int_0^{\frac{\pi}{4}}\tan^2(x)\sqrt{\tan x}\,dx$$

$$=\left[\frac{2}{3}\tan^{\frac{3}{2}}(x)\right]_0^{\frac{\pi}{4}}-\int_0^{\frac{\pi}{4}}\tan^2(x)\sqrt{\tan x}\,dx$$

$$A=\frac{2}{3}-\int_0^{\frac{\pi}{4}}\tan^2(x)\sqrt{\tan x}\,dx$$

نضع : $I=\int_0^{\frac{\pi}{4}}\tan^2(x)\sqrt{\tan x}\,dx$.

نستعمل المكاملة بتغييرالمتغير؛ بوضع $t=\sqrt{\tan x}$

لدينا : $x=0\Rightarrow t=0$ و $x=\frac{\pi}{4}\Rightarrow t=1$

و $dt=\left(\sqrt{\tan x}\right)'dx=\frac{1+\tan^2(x)}{2\sqrt{\tan x}}dx=\frac{1+t^4}{2t}dx$

إذن : $dx=\frac{2t}{1+t^4}dt$. ومنه نستنتج أن :

$$I=\int_0^1 t^4 t.\frac{2t}{1+t^4}dt=2\int_0^1\frac{t^6}{1+t^4}dt=2\int_0^1\frac{t^6+t^2-t^2}{1+t^4}dt$$

$$=2\int_0^1\left(t^2-\frac{t^2}{1+t^4}\right)dt=2\left(\left[\frac{1}{3}t^3\right]_0^1-\int_0^1\frac{t^2}{1+t^4}dt\right)$$

$$I=\frac{2}{3}-2\int_0^1\frac{t^2}{1+t^4}dt$$

لنحسب التكامل : $J=\int_0^1\frac{t^2}{1+t^4}dt$.

لدينا : $\frac{t^2}{1+t^4}=\frac{t^2}{\left(t^2+\sqrt{2}t+1\right)\left(t^2-\sqrt{2}t+1\right)}$

$$=-\frac{\sqrt{2}}{4}\left(\frac{t}{t^2+\sqrt{2}t+1}-\frac{t}{t^2-\sqrt{2}t+1}\right)$$

$$\frac{t^2}{1+t^4}=-\frac{\sqrt{2}}{8}\left(\frac{2t+\sqrt{2}-\sqrt{2}}{t^2+\sqrt{2}t+1}-\frac{2t-\sqrt{2}+\sqrt{2}}{t^2-\sqrt{2}t+1}\right)$$

$$=-\frac{\sqrt{2}}{8}\left(\frac{\left(t^2+\sqrt{2}t+1\right)'}{t^2+\sqrt{2}t+1}-\frac{\sqrt{2}}{\frac{1}{2}+\left(t+\frac{\sqrt{2}}{2}\right)^2}-\frac{\left(t^2-\sqrt{2}t+1\right)'}{t^2-\sqrt{2}t+1}-\frac{\sqrt{2}}{\frac{1}{2}+\left(t-\frac{\sqrt{2}}{2}\right)^2}\right)$$

$$=-\frac{\sqrt{2}}{8}\left(\frac{\left(t^2+\sqrt{2}t+1\right)'}{t^2+\sqrt{2}t+1}-2\frac{\sqrt{2}}{1+\left(\sqrt{2}t+1\right)^2}-\frac{\left(t^2-\sqrt{2}t+1\right)'}{t^2-\sqrt{2}t+1}-2\frac{\sqrt{2}}{1+\left(\sqrt{2}t-1\right)^2}\right)$$

$$=-\frac{\sqrt{2}}{8}\left(\frac{\left(t^2+\sqrt{2}t+1\right)'}{t^2+\sqrt{2}t+1}-2\frac{\left(\sqrt{2}t+1\right)'}{1+\left(\sqrt{2}t+1\right)^2}-\frac{\left(t^2-\sqrt{2}t+1\right)'}{t^2-\sqrt{2}t+1}-2\frac{\left(\sqrt{2}t-1\right)'}{1+\left(\sqrt{2}t-1\right)^2}\right)$$

ومنه فإن : $J=\int_0^1\frac{t^2}{1+t^4}dt=-\frac{\sqrt{2}}{8}\left[\ln\left(\frac{t^2+\sqrt{2}t+1}{t^2-\sqrt{2}t+1}\right)\right]_0^1$

$$+\frac{\sqrt{2}}{4}\left[Arc\tan\left(\sqrt{2}t+1\right)+Arc\tan\left(\sqrt{2}t-1\right)\right]_0^1$$

$$J=-\frac{\sqrt{2}}{8}\ln\left(\frac{2+\sqrt{2}}{2-\sqrt{2}}\right)$$

$$+\frac{\sqrt{2}}{4}\left(Arc\tan\left(\sqrt{2}+1\right)-\frac{\pi}{4}+Arc\tan\left(\sqrt{2}-1\right)+\frac{\pi}{4}\right)$$

ولدينا : $\frac{2+\sqrt{2}}{2-\sqrt{2}}=\frac{\left(2+\sqrt{2}\right)^2}{4-2}=\frac{6+4\sqrt{2}}{2}=3+2\sqrt{2}$.

و : $\sqrt{2}-1=\frac{1}{\sqrt{2}+1}$

و : $\boxed{\forall x>0\ :\ Arc\tan(x)+Arc\tan\left(\frac{1}{x}\right)=\frac{\pi}{2}}$

إذن : $Arc\tan\left(\sqrt{2}+1\right)+Arc\tan\left(\sqrt{2}-1\right)=\dfrac{\pi}{2}$

ومنه فإن : $J=-\dfrac{\sqrt{2}}{8}\ln\left(3+2\sqrt{2}\right)+\dfrac{\sqrt{2}}{4}\left(\dfrac{\pi}{2}-\dfrac{\pi}{4}+\dfrac{\pi}{4}\right)$

$$J=-\frac{\sqrt{2}}{8}\ln\left(3+2\sqrt{2}\right)+\frac{\pi\sqrt{2}}{8}$$

إذن : $I=\dfrac{2}{3}-2\left(-\dfrac{\sqrt{2}}{8}\ln\left(3+2\sqrt{2}\right)+\dfrac{\pi\sqrt{2}}{8}\right)$

$$I=\frac{2}{3}+\frac{\sqrt{2}}{4}\ln\left(3+2\sqrt{2}\right)-\frac{\pi\sqrt{2}}{4}$$

وبالتالي فإن:

$$A=\frac{2}{3}-I=\frac{2}{3}-\left(\frac{2}{3}+\frac{\sqrt{2}}{4}\ln\left(3+2\sqrt{2}\right)-\frac{\pi\sqrt{2}}{4}\right)$$

$$A=\int_0^{\frac{\pi}{4}}\sqrt{\tan x}\,dx=\boxed{-\frac{\sqrt{2}}{4}\ln\left(3+2\sqrt{2}\right)+\frac{\pi\sqrt{2}}{4}}$$

التمرين 2 :

أحسب التكامل التالي : $B=\displaystyle\int_0^1\frac{1}{1+x^4}dx$.

الجواب

لدينا : $\dfrac{1}{1+x^4}=\dfrac{1}{\left(x^2+\sqrt{2}x+1\right)\left(x^2-\sqrt{2}x+1\right)}$

$$=\left(\frac{\frac{\sqrt{2}}{4}x+\frac{1}{2}}{x^2+\sqrt{2}x+1}+\frac{-\frac{\sqrt{2}}{4}x+\frac{1}{2}}{x^2-\sqrt{2}x+1}\right)$$

$$=\left(\frac{\frac{\sqrt{2}}{8}\left(2x+\sqrt{2}\right)+\frac{1}{4}}{x^2+\sqrt{2}x+1}+\frac{-\frac{\sqrt{2}}{8}\left(2x-\sqrt{2}\right)+\frac{1}{4}}{x^2-\sqrt{2}x+1}\right)$$

$$=\frac{\sqrt{2}}{8}\frac{\left(x^2+\sqrt{2}x+1\right)'}{x^2+\sqrt{2}x+1}-\frac{\sqrt{2}}{8}\frac{\left(x^2-\sqrt{2}x+1\right)'}{x^2-\sqrt{2}x+1}+\frac{\frac{1}{4}}{\left(x+\frac{\sqrt{2}}{2}\right)^2+\frac{1}{2}}+\frac{\frac{1}{4}}{\left(x-\frac{\sqrt{2}}{2}\right)^2+\frac{1}{2}}$$

$$=\frac{\sqrt{2}}{8}\frac{\left(x^2+\sqrt{2}x+1\right)'}{x^2+\sqrt{2}x+1}-\frac{\sqrt{2}}{8}\frac{\left(x^2-\sqrt{2}x+1\right)'}{x^2-\sqrt{2}x+1}+\frac{\frac{1}{2}}{1+\left(\sqrt{2}x+1\right)^2}+\frac{\frac{1}{2}}{1+\left(\sqrt{2}x-1\right)^2}$$

$$\frac{1}{1+x^4}=\frac{\sqrt{2}}{8}\frac{\left(x^2+\sqrt{2}x+1\right)'}{x^2+\sqrt{2}x+1}-\frac{\sqrt{2}}{8}\frac{\left(x^2-\sqrt{2}x+1\right)'}{x^2-\sqrt{2}x+1}+\frac{\sqrt{2}}{2}\frac{\left(\sqrt{2}x+1\right)'}{1+\left(\sqrt{2}x+1\right)^2}+\frac{\sqrt{2}}{2}\frac{\left(\sqrt{2}x-1\right)'}{1+\left(\sqrt{2}x-1\right)^2}$$

$$B=\int_0^1\frac{1}{1+x^4}dx=\frac{\sqrt{2}}{8}\left[\ln\left(\frac{x^2+\sqrt{2}x+1}{x^2-\sqrt{2}x+1}\right)\right]_0^1+\frac{\sqrt{2}}{2}\left[Arc\tan\left(\sqrt{2}x+1\right)+Arc\tan\left(\sqrt{2}x-1\right)\right]_0^1$$

$$B=\boxed{\frac{\sqrt{2}}{8}\ln\left(\frac{2+\sqrt{2}}{2-\sqrt{2}}\right)}+\frac{\sqrt{2}}{2}\left(Arc\tan\left(\sqrt{2}+1\right)-\frac{\pi}{4}+Arc\tan\left(\sqrt{2}-1\right)+\frac{\pi}{4}\right)$$

كما رأينا في السؤال السابق ؛ نحصل على ما يلي :

$$B=\int_0^1\frac{1}{1+x^4}dx=\boxed{\frac{\sqrt{2}}{8}\ln\left(3+2\sqrt{2}\right)+\frac{\pi\sqrt{2}}{4}}$$

التمرين 3 :

أحسب التكامل التالي : $C=\displaystyle\int_0^1\sqrt{1+x^2}dx$.

الجواب

نضع : $\begin{cases}u'(x)=1\\ v(x)=\sqrt{1+x^2}\end{cases}$. إذن : $\begin{cases}u(x)=x\\ v'(x)=\dfrac{x}{\sqrt{1+x^2}}\end{cases}$.

لدينا u و v قابلتين للإشتقاق على المجال $[0,1]$ و u' و v' متصلتين على المجال $[0,1]$. حسب صيغة المكاملة بالأجزاء ؛ لدينا :

$$C=\left[u(x)\times v(x)\right]_0^1-\int_0^1 u(x)\times v'(x)dx$$

$$=\left[x\sqrt{1+x^2}\right]_0^1-\int_0^1\frac{x^2}{\sqrt{1+x^2}}dx$$

$$=\sqrt{2}-\int_0^1\frac{1+x^2-1}{\sqrt{1+x^2}}dx$$

$$=\sqrt{2}-\int_0^1\sqrt{1+x^2}dx+\int_0^1\frac{1}{\sqrt{1+x^2}}dx$$

$$C=\sqrt{2}-C+\int_0^1\frac{1}{\sqrt{1+x^2}}dx$$

إذن : $2C=\sqrt{2}-\displaystyle\int_0^1\frac{1}{\sqrt{1+x^2}}dx$

ومنه فإن : $C=\dfrac{1}{2}\sqrt{2}-\dfrac{1}{2}\displaystyle\int_0^1\frac{1}{\sqrt{1+x^2}}dx$

نضع : $I=\int_0^1 \frac{1}{\sqrt{1+x^2}}dx$.

نعتبر المتغير : $t=x+\sqrt{1+x^2}$.

الدالة $x\mapsto x+\sqrt{1+x^2}$ قابلة للإشتقاق على المجال $[0,1]$ ومشتقتها متصلة على المجال $[0,1]$.

لدينا : $\begin{cases} x=0\Rightarrow t=1 \\ x=1\Rightarrow t=1+\sqrt{2} \end{cases}$ و $dt=\left(x+\sqrt{1+x^2}\right)'dx$

$$=\left(1+\frac{\cancel{2}x}{\cancel{2}\sqrt{1+x^2}}\right)dx$$

$$=\frac{x+\sqrt{1+x^2}}{\sqrt{1+x^2}}dx$$

$$dt=\frac{t}{\sqrt{1+x^2}}dx$$

ومنه نستنتج أن : $\frac{dx}{\sqrt{1+x^2}}=\frac{dt}{t}$.

وحسب المكاملة بتغيير المتغير ؛ لدينا :

$$I=\int_1^{1+\sqrt{2}}\frac{dt}{t}=\left[\ln|t|\right]_1^{1+\sqrt{2}}=\boxed{\ln\left(1+\sqrt{2}\right)}$$

ومنه نجد : $C=\frac{1}{2}\sqrt{2}-\frac{1}{2}\ln\left(1+\sqrt{2}\right)$

$$\int_0^1\frac{1}{\sqrt{1+x^2}}dx=\boxed{\frac{1}{2}\left(\sqrt{2}-\ln\left(1+\sqrt{2}\right)\right)}$$

☜ **ملاحظة :** يمكن حساب التكامل I بطرق أخرى .

مثلا: نضع المتغير : $x=\tan(t)$ أي $t=Arc\tan(x)$

لدينا : $\begin{cases} x=0\Rightarrow t=0 \\ x=1\Rightarrow t=\frac{\pi}{4} \end{cases}$

و : $dx=\tan'(t)dt=\left(1+\tan^2(t)\right)dt$

الدالة $Arc\tan$ قابلة للإشتقاق على المجال $[0,1]$ و مشتقتها متصلة على المجال $[0,1]$. حسب المكاملة بتغيير المتغير ؛ لدينا :

$$I=\int_0^{\frac{\pi}{4}}\frac{1}{\sqrt{1+\tan^2(t)}}\left(1+\tan^2(t)\right)dt=\int_0^{\frac{\pi}{4}}\sqrt{1+\tan^2(t)}dt$$

$$I=\int_0^{\frac{\pi}{4}}\sqrt{\frac{1}{\cos^2(t)}}dt=\int_0^{\frac{\pi}{4}}\left|\frac{1}{\cos(t)}\right|dt$$

وبما أن : $\cos(x)>0$: $\forall t\in\left[0,\frac{\pi}{4}\right]$ ؛ فإن :

$$I=\int_0^{\frac{\pi}{4}}\frac{1}{\cos(t)}dt$$

نعتبر المتغير $u=\tan\left(\frac{t}{2}\right)$. إذن :

$$\begin{cases} t=0\Rightarrow u=0 \\ t=\frac{\pi}{4}\Rightarrow u=\tan\left(\frac{\pi}{8}\right) \end{cases}$$

ولدينا : $t\mapsto\tan\left(\frac{t}{2}\right)$ قابلة للإشتقاق على المجال $\left[0,\frac{\pi}{4}\right]$ ومشتقتها متصلة على المجال $\left[0,\frac{\pi}{4}\right]$.

و : $du=\left(\tan\left(\frac{t}{2}\right)\right)'dt=\frac{1}{2}\left(1+\tan^2\left(\frac{t}{2}\right)\right)dt$

$$\Rightarrow du=\frac{1}{2}\left(1+u^2\right)dt\Rightarrow dt=\frac{2du}{1+u^2}$$

و : $\cos(t)=\frac{1-u^2}{1+u^2}$.

وحسب المكاملة بتغيير المتغير ؛ لدينا :

$$I=\int_0^{\tan\left(\frac{\pi}{8}\right)}\frac{1+u^2}{1-u^2}\times\frac{2u}{1+u^2}du=\int_0^{\tan\left(\frac{\pi}{8}\right)}\frac{2u}{1-u^2}du$$

$$I=\int_0^{\tan\left(\frac{\pi}{8}\right)}\left(\frac{1}{1+u}+\frac{1}{1-u}\right)du=\int_0^{\tan\left(\frac{\pi}{8}\right)}\left(\frac{(1+u)'}{1+u}-\frac{(1-u)'}{1-u}\right)du$$

$$I=\left[\ln|1+u|-\ln|1-u|\right]_0^{\tan\left(\frac{\pi}{8}\right)}=\left[\ln\left|\frac{1+u}{1-u}\right|\right]_0^{\tan\left(\frac{\pi}{8}\right)}$$

☜ لنحدد قيمة العدد : $\tan\left(\frac{\pi}{8}\right)$.

لدينا : $1=\tan\left(\frac{\pi}{4}\right)=\tan\left(2\times\frac{\pi}{8}\right)=\frac{2\tan\left(\frac{\pi}{8}\right)}{1-\tan^2\left(\frac{\pi}{8}\right)}$

إذن : $\tan^2\left(\frac{\pi}{8}\right)+2\tan\left(\frac{\pi}{8}\right)-1=0$.

وبوضع $t=\tan\left(\frac{\pi}{8}\right)$ ؛ نجد : $t^2+2t-1=0$.

لدينا $\Delta'=2$ إذن : $t=\frac{-b'+\sqrt{\Delta'}}{a}=-1+\sqrt{2}$

أو : $t=\frac{-b'-\sqrt{\Delta'}}{a}=-1-\sqrt{2}=-\left(1+\sqrt{2}\right)$

وبما أن $\frac{\pi}{8}\in\left]0,\frac{\pi}{2}\right[\Rightarrow t=\tan\left(\frac{\pi}{8}\right)>0$ ؛ فإن :

$$\boxed{\tan\left(\frac{\pi}{8}\right)=-1+\sqrt{2}}$$

إذن :
$$I=\left[\ln\left|\frac{1+u}{1-u}\right|\right]_0^{-1+\sqrt{2}}=\ln\left(\frac{\sqrt{2}}{2-\sqrt{2}}\right)$$
$$I=\ln\left(\frac{\sqrt{2}\left(2+\sqrt{2}\right)}{2}\right)=\ln\left(1+\sqrt{2}\right)$$

ومنه فإن : $\int_0^1\frac{1}{\sqrt{1+x^2}}dx=\boxed{\ln\left(1+\sqrt{2}\right)}$.

تطبيق : بوضع $t=x-\frac{1}{x}$ ؛ أحسب التكامل التالي:
$$J=\int_1^2\frac{x^2+1}{x\sqrt{x^4-x^2+1}}.dx$$

لدينا :
$$\begin{cases}x=1\Rightarrow t=0\\ x=2\Rightarrow t=2-\frac{1}{2}=\frac{3}{2}\end{cases}$$

الدالة $x\mapsto x-\frac{1}{x}$ قابلة للإشتقاق على المجال $[1,2]$ ومشتقتها متصلة على المجال $[1,2]$.

ولدينا : $dt=\left(x-\frac{1}{x}\right)'dx=\left(1+\frac{1}{x^2}\right)dx=\frac{x^2+1}{x^2}dx$

و لدينا : $t=x-\frac{1}{x}\Rightarrow t^2=\left(x-\frac{1}{x}\right)^2=x^2+\frac{1}{x^2}-2$

إذن : $x^2+\frac{1}{x^2}=t-2$.

ومنه فإن :
$$\frac{x^2+1}{x\sqrt{x^4-x^2+1}}=\frac{x^2+1}{x^2}\times\frac{1}{\frac{1}{x}\sqrt{x^4-x^2+1}}$$
$$=\frac{x^2+1}{x^2}\times\frac{1}{\sqrt{x^2+\frac{1}{x^2}-1}}$$

حسب المكاملة بتغيير المتغير ؛ نجد :
$$J=\int_0^{\frac{3}{2}}\frac{dt}{\sqrt{t^2+2-1}}=\int_0^{\frac{3}{2}}\frac{dt}{\sqrt{t^2+1}}=\ln\left(1+\sqrt{2}\right)$$

وبالتالي فإن : $\int_1^2\frac{x^2+1}{x\sqrt{x^4-x^2+1}}.dx=\boxed{\ln\left(1+\sqrt{2}\right)}$.

التمرين 4 :

أحسب التكامل التالي : $D=\int_0^{\frac{\pi}{3}}\frac{1}{\cos^3(x)}dx$.

الجواب

لدينا : $D=\int_0^{\frac{\pi}{3}}\frac{1}{\cos(x)}\times\frac{1}{\cos^2(x)}dx=\int_0^{\frac{\pi}{3}}\frac{1}{\cos(x)}\times\tan'(x).dx$

لدينا $\tan$ و $x\mapsto\frac{1}{\cos(x)}$ قابلتين للإشتقاق على المجال $\left[0,\frac{\pi}{3}\right]$ والتيها المشتقة متصلتين على المجال $\left[0,\frac{\pi}{3}\right]$. حسب المكاملة بالأجزاء ؛ لدينا :
$$D=\left[\frac{1}{\cos(x)}\times\tan(x)\right]_0^{\frac{\pi}{3}}-\int_0^{\frac{\pi}{3}}\left(\frac{1}{\cos(x)}\right)'\tan(x)dx$$
$$D=\left[\sin(x)\right]_0^{\frac{\pi}{3}}-\int_0^{\frac{\pi}{3}}-\frac{\cos'(x)}{\cos^2(x)}\tan(x)dx$$
$$D=\left[\sin(x)\right]_0^{\frac{\pi}{3}}-\int_0^{\frac{\pi}{3}}\frac{\sin(x)}{\cos^2(x)}\tan(x)dx$$
$$D=\left[\sin(x)\right]_0^{\frac{\pi}{3}}-\int_0^{\frac{\pi}{3}}\frac{\sin^2(x)}{\cos^3(x)}dx$$
$$D=\left[\sin(x)\right]_0^{\frac{\pi}{3}}-\int_0^{\frac{\pi}{3}}\frac{1-\cos^2(x)}{\cos^3(x)}dx$$
$$D=\left[\sin(x)\right]_0^{\frac{\pi}{3}}-\int_0^{\frac{\pi}{3}}\frac{1}{\cos^3(x)}dx+\int_0^{\frac{\pi}{3}}\frac{\cos^2(x)}{\cos^3(x)}dx$$
$$D=\left[\sin(x)\right]_0^{\frac{\pi}{3}}-D+\int_0^{\frac{\pi}{3}}\frac{1}{\cos(x)}dx$$

إذن : $2D=\frac{\sqrt{3}}{2}+\int_0^{\frac{\pi}{3}}\frac{1}{\cos(x)}dx$

نعتبر المتغير $t=\tan\left(\frac{x}{2}\right)$. إذن :

$x=0\Rightarrow t=0$ و $x=\frac{\pi}{3}\Rightarrow t=\tan\left(\frac{\pi}{6}\right)=\frac{\sqrt{3}}{3}$

ولدينا : $x\mapsto\tan\left(\frac{x}{2}\right)$ قابلة للإشتقاق على المجال $\left[0,\frac{\pi}{3}\right]$ ومشتقتها متصلة على المجال $\left[0,\frac{\pi}{3}\right]$.

و $dt=\frac{1}{2}\left(1+t^2\right)dx\Rightarrow dx=\frac{2dt}{1+t^2}$ و $\cos(x)=\frac{1-t^2}{1+t^2}$.

وحسب المكاملة بتغيير المتغير ؛ لدينا :
$$\int_0^{\frac{\pi}{3}}\frac{1}{\cos(x)}dx=\int_0^{\frac{\sqrt{3}}{3}}\frac{1+t^2}{1-t^2}\times\frac{2t}{1+t^2}du=\int_0^{\frac{\sqrt{3}}{3}}\frac{2t}{1-t^2}dt$$
$$=\left[\ln\left|\frac{1+t}{1-t}\right|\right]_0^{\frac{\sqrt{3}}{3}}=\ln\left(2+\sqrt{3}\right)$$

وبالتالي فإن : $\int_0^{\frac{\pi}{3}}\frac{1}{\cos^3(x)}dx=\boxed{\frac{\sqrt{3}}{4}+\frac{1}{2}\ln\left(2+\sqrt{3}\right)}$.

التمرين 5 :

أحسب التكامل التالي: $E=\int_0^{\frac{\pi}{4}}\sqrt{1+\tan(x)+\tan^2(x)}.dx$

الجواب

لدينا : $E=\int_0^{\frac{\pi}{4}}\frac{1+\tan(x)+\tan^2(x)}{\sqrt{1+\tan(x)+\tan^2(x)}}.dx$

$$E=\int_0^{\frac{\pi}{4}}\frac{1+\tan^2(x)}{\sqrt{1+\tan(x)+\tan^2(x)}}.dx+\int_0^{\frac{\pi}{4}}\frac{\tan(x)}{\sqrt{1+\tan(x)+\tan^2(x)}}.dx$$

نضع : $I=\int_0^{\frac{\pi}{4}}\frac{1+\tan^2(x)}{\sqrt{1+\tan(x)+\tan^2(x)}}.dx$

و : $J=\int_0^{\frac{\pi}{4}}\frac{\tan(x)}{\sqrt{1+\tan(x)+\tan^2(x)}}.dx$

إذن : $E=I+J$

حساب I :

نضع : $t=\tan(x)$. إذن : و لدينا : $\begin{cases}x=0\Rightarrow t=0\\ x=\frac{\pi}{4}\Rightarrow t=1\end{cases}$

$$dt=\tan'(x)dx=\left(1+\tan^2(x)\right)dx=\left(1+t^2\right)dx$$

$$\Rightarrow dx=\frac{dt}{1+t^2}$$

الدالة $\tan$ قابلة للاشتقاق على المجال $\left[0,\frac{\pi}{4}\right]$ و مشتقتها متصلة على المجال $\left[0,\frac{\pi}{4}\right]$. حسب المكاملة بتغيير المتغير ؛ نحصل على :

$$I=\int_0^1\frac{1+t^2}{\sqrt{1+t+t^2}}\times\frac{dt}{1+t^2}=\int_0^1\frac{dt}{\sqrt{1+t+t^2}}$$

$$I=\int_0^1\frac{dt}{\sqrt{\left(t+\frac{1}{2}\right)^2+\frac{3}{4}}}=\int_0^1\frac{dt}{\sqrt{\frac{3}{4}\left(\frac{4}{3}\left(t+\frac{1}{2}\right)^2+1\right)}}$$

$$I=\int_0^1\frac{\sqrt{\frac{4}{3}}.dt}{\sqrt{\left(\sqrt{\frac{4}{3}}\left(t+\frac{1}{2}\right)\right)^2+1}}$$

نضع : $u=\sqrt{\frac{4}{3}}\left(t+\frac{1}{2}\right)$. إذن : $du=\sqrt{\frac{4}{3}}.dt$ ؛ و لدينا:

$$\begin{cases}t=0\Rightarrow u=\frac{\sqrt{3}}{3}\\ t=1\Rightarrow u=\sqrt{3}\end{cases}$$

الدالة $t\mapsto\sqrt{\frac{4}{3}}\left(t+\frac{1}{2}\right)$ قابلة للاشتقاق على المجال $[0,1]$ ومشتقتها متصلة على المجال $[0,1]$. حسب تقنية المكاملة بالأجزاء ؛ نجد : $I=\int_{\sqrt{3}}^{\frac{\sqrt{3}}{3}}\frac{du}{\sqrt{1+u^2}}$.

وبوضع : $x=u+\sqrt{1+u^2}$ ؛ نجد :

$$\begin{cases}u=\sqrt{3}\Rightarrow x=\sqrt{3}+2\\ u=\frac{\sqrt{3}}{3}\Rightarrow x=\sqrt{3}\end{cases}$$

الدالة $u\mapsto u+\sqrt{1+u^2}$ قابلة للاشتقاق على المجال $\left[\sqrt{3},\frac{\sqrt{3}}{3}\right]$ ومشتقتها متصلة على المجال $\left[\sqrt{3},\frac{\sqrt{3}}{3}\right]$.

ولدينا : $dx=\left(u+\sqrt{1+u^2}\right)'du=\frac{u+\sqrt{1+u^2}}{\sqrt{1+u^2}}du$

$$\Rightarrow\frac{dx}{x}=\frac{du}{\sqrt{1+u^2}}$$

حسب تقنية المكاملة بالأجزاء ؛ نجد :

$$I=\int_{\sqrt{3}}^{\sqrt{3}+2}\frac{dx}{x}=\left[\ln|x|\right]_{\sqrt{3}}^{\sqrt{3}+2}=\boxed{\ln\left(1+\frac{2}{\sqrt{3}}\right)}$$

حساب J :

لدينا : $J=\int_0^{\frac{\pi}{4}}\frac{\sin(x)}{\sqrt{1+\sin(x)\cos(x)}}.dx$

لأن : $\tan(x)=\frac{\sin(x)}{\cos(x)}$ و $1+\tan^2(x)=\frac{1}{\cos^2(x)}$.

نضع : $t=\frac{\pi}{4}-x$. إذن : $dt=-dx$ و لدينا :

$$\sin(x)\cos(x)=\frac{1}{2}\sin(2x)=\frac{1}{2}\sin\left(2\left(\frac{\pi}{4}-t\right)\right)=\frac{1}{2}\cos(2t)$$

$$\sin(x)=\sin\left(\frac{\pi}{4}-t\right)=\frac{\sqrt{2}}{2}(\cos(t)+\sin(t))$$

ومنه فإن : $J=\frac{\sqrt{2}}{2}\int_0^{\frac{\pi}{4}}\frac{\cos(t)+\sin(t)}{\sqrt{1+\frac{1}{2}\cos(2t)}}.dt$

وبما أن: $\cos(2t)=2\cos^2(t)-1$ و $\cos(2t)=1-2\sin^2(t)$

فإن: $J=\frac{\sqrt{2}}{2}\int_0^{\frac{\pi}{4}}\frac{\cos(t)}{\sqrt{\frac{3}{2}-\sin^2(t)}}.dt-\frac{\sqrt{2}}{2}\int_0^{\frac{\pi}{4}}\frac{-\sin(t)}{\sqrt{\frac{1}{2}+\cos^2(t)}}.dt$

نضع : $J_1=\int_0^{\frac{\pi}{4}}\frac{\cos(t)}{\sqrt{\frac{3}{2}-\sin^2(t)}}.dt$ و $J_2=\int_0^{\frac{\pi}{4}}\frac{-\sin(t)}{\sqrt{\frac{1}{2}+\cos^2(t)}}.dt$

☜ لدينا : $$J_1 = \int_0^{\frac{\pi}{4}} \frac{\sqrt{\frac{2}{3}}\cos(t)}{\sqrt{\left(1-\left(\sqrt{\frac{2}{3}}\sin(t)\right)^2\right)}}.dt$$

نضع : $x=\sqrt{\frac{2}{3}}\sin(t)$. إذن : $dx=\sqrt{\frac{2}{3}}\cos(t)dt$.

و $\begin{cases} t=0\Rightarrow x=0 \\ t=\frac{\pi}{4}\Rightarrow x=\frac{1}{\sqrt{3}}\end{cases}$. حسب المكاملة بتغيير المتغير ؛

نجد أن : $J_1=\int_0^{\frac{1}{\sqrt{3}}}\frac{dx}{\sqrt{1-x^2}}=\left[Arc\sin(x)\right]_0^{\frac{1}{\sqrt{3}}}=Arc\sin\left(\frac{1}{\sqrt{3}}\right)$

☜ نضع : $x=\cos(t)$. إذن : $dx=-\sin(t)dt$.

و : $\begin{cases} t=0\Rightarrow x=1\\ t=\frac{\pi}{4}\Rightarrow x=\frac{\sqrt{2}}{2}\end{cases}$

حسب المكاملة بالأجزاء ؛ نحصل على :

$$J_2=\int_1^{\frac{\sqrt{2}}{2}}\frac{dx}{\sqrt{\frac{1}{2}+x^2}}=\int_1^{\frac{\sqrt{2}}{2}}\frac{\sqrt{2}dx}{\sqrt{1+\left(\sqrt{2}x\right)^2}}$$

وبوضع : $t=\sqrt{2}x$ ؛ لدينا : $dt=\sqrt{2}dx$. إذن :

$$J_2=\int_{\sqrt{2}}^{1}\frac{dt}{\sqrt{1+t^2}}=\left[\ln\left(t+\sqrt{1+t^2}\right)\right]_{\sqrt{2}}^{1}$$

$$J_2=\boxed{\ln\left(1+\sqrt{2}\right)-\ln\left(\sqrt{2}+\sqrt{3}\right)}$$

Le fameux truc

ومنه فإن : $$J=\frac{\sqrt{2}}{2}J_1+\frac{\sqrt{2}}{2}J_2$$

$$=\frac{\sqrt{2}}{2}Arc\sin\left(\frac{1}{\sqrt{3}}\right)+\frac{\sqrt{2}}{2}\ln\left(\frac{1+\sqrt{2}}{\sqrt{2}+\sqrt{3}}\right)$$

و بالتالي فإن : $E=I+J$

$$E=\ln\left(1+\frac{2}{\sqrt{3}}\right)+\frac{\sqrt{2}}{2}Arc\sin\left(\frac{1}{\sqrt{3}}\right)+\frac{\sqrt{2}}{2}\ln\left(\frac{1+\sqrt{2}}{\sqrt{2}+\sqrt{3}}\right)$$

التمرين 6 :

أحسب التكامل التالي: $F=\int_0^{\frac{\pi}{4}}\ln\left(1+\tan(x)\right).dx$.

الجواب

لدينا : $$F=\int_0^{\frac{\pi}{4}}\ln\left(\frac{\cos(x)+\sin(x)}{\cos(x)}\right).dx$$

$$F=\int_0^{\frac{\pi}{4}}\ln\left(\cos(x)+\sin(x)\right).dx-\int_0^{\frac{\pi}{4}}\ln\left(\cos(x)\right).dx$$

$$F=\int_0^{\frac{\pi}{4}}\ln\left(\sqrt{2}\sin\left(x+\frac{\pi}{4}\right)\right).dx-\int_0^{\frac{\pi}{4}}\ln\left(\cos(x)\right).dx$$

$$F=\frac{\pi}{4}\ln\left(\sqrt{2}\right)+\int_0^{\frac{\pi}{4}}\ln\left(\sin\left(x+\frac{\pi}{4}\right)\right).dx-\int_0^{\frac{\pi}{4}}\ln\left(\cos(x)\right).dx$$

نضع : $t=x+\frac{\pi}{4}$. إذن : $dt=dx$. ولدينا :

$$\begin{cases} x=0\Rightarrow t=\frac{\pi}{4}\\ x=\frac{\pi}{4}\Rightarrow t=\frac{\pi}{2}\end{cases}$$

إذن : $$\int_0^{\frac{\pi}{4}}\ln\left(\sin\left(x+\frac{\pi}{4}\right)\right).dx=\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\ln\left(\sin(t)\right).dt$$

و : $$\int_0^{\frac{\pi}{4}}\ln\left(\cos(x)\right).dx=\int_0^{\frac{\pi}{4}}\ln\left(\sin\left(\frac{\pi}{2}-x\right)\right).dx$$

نضع : $t=\frac{\pi}{2}-x$. إذن : $dt=-dx$. ولدينا :

$$\begin{cases} x=0\Rightarrow t=\frac{\pi}{2}\\ x=\frac{\pi}{4}\Rightarrow t=\frac{\pi}{4}\end{cases}$$

On pourra retenir l'idée qu'il y a toujours de multiples manières d'intégrer par parties , et qu'un choix judicieux peut simplifier considérablement les calculs.

ومنه فإن : $$\int_0^{\frac{\pi}{4}}\ln\left(\cos(x)\right).dx=-\int_{\frac{\pi}{2}}^{\frac{\pi}{4}}\ln\left(\sin(t)\right).dt$$

وبالتالي فإن : $F=\frac{\pi}{4}\ln\left(\sqrt{2}\right)=\frac{\pi}{8}\ln(2)$.

$$\int_0^{\frac{\pi}{4}}\ln\left(1+\tan(x)\right)dx=\boxed{\frac{\pi}{8}\ln(2)}$$

التمرين 7 :

$I_n=\int_0^1\frac{dx}{\left(1+x^2\right)^n}$. حدد العلاقة بين I_n و I_{n+1} . $(n\in\mathbb{N})$.

الجواب

نضع: $\begin{cases} u(x)=x\\ v'(x)=\frac{1}{\left(1+x^2\right)^n}\end{cases}$ إذن : $\begin{cases} u'(x)=1\\ v(x)=-2n\frac{x}{\left(1+x^2\right)^{n+1}}\end{cases}$

$$I_n=\left[\frac{x}{\left(1+x^2\right)^n}\right]_0^1+2n\int_0^1\frac{x^2}{\left(1+x^2\right)^{n+1}}dx$$

$$....I_n=\frac{1}{2^n}+2n\int_0^1\frac{1+x^2-1}{\left(1+x^2\right)^{n+1}}dx=\frac{1}{2^n}+2n\left(I_n-I_{n+1}\right)$$