تصحيح الفرض المحروس رقم 1

الدورة الثانية

تمرين 01 : $g_n(x) = n.\ln(x) - \frac{1}{x}$

$\lim\limits_{x\to+\infty} g_n(x) = \lim\limits_{+\infty} n\ln(x) - \frac{1}{x}$

$\lim\limits_{+\infty} g_n(x) = \boxed{+\infty}$

$\lim\limits_{0^+} g_n(x) = \lim\limits_{0^+} n\ln(x) - \frac{1}{x}$

$\lim\limits_{0^+} g_n(x) = \boxed{-\infty}$

② $g_n'(x) = \frac{n}{x} + \frac{1}{x^2} > 0$

x	0	$+\infty$
$g_n(x)$	$-\infty$ ↗	$+\infty$

* بين أن $\exists! \alpha_n \in \mathbb{R}^{*+} / g_n(\alpha_n)=0$

- الدالة g_n متصلة وتزايدية قطعا على $\mathbb{R}^{+*}$، ومنه $g_n(x)$ تقابل من $\mathbb{R}^+$ نحو $\mathbb{R}$ و $0 \in \mathbb{R}$

إذن: $\boxed{\exists! \alpha_n \in \mathbb{R}^{+*} / g_n(\alpha_n)=0}$

- على المجال $]0,\alpha_n[$

$\left.\begin{array}{l} 0<x<\alpha_n \\ g_n \text{ تزايدية} \end{array}\right\} \Rightarrow g_n(x)<g_n(\alpha_n)$

$\Rightarrow g_n(x)<0$

- على المجال $]\alpha_n,+\infty[$

$\left.\begin{array}{l} x>\alpha_n \\ g_n \text{ تزايدية} \end{array}\right\} \Rightarrow g_n(x)>g_n(\alpha_n)$

$\Rightarrow g_n(x)>0$

x		α_n	
$g_n(x)$	$-$	0	$+$

③ لدينا $g_n(\alpha_n)=0$ و $g_n(1)<0$

$g_n(1)<g_n(\alpha_n)$

و g_n تزايدية : $\boxed{1<\alpha_n}$

* $g_n(\alpha_n)=0 \iff \ln\alpha_n = \frac{1}{n.\alpha_n}$

$\alpha_n > 1 \Rightarrow \alpha_n.n > n$

ومنه $\frac{1}{\alpha_n.n} < \frac{1}{n}$

إذن $\ln\alpha_n < \frac{1}{n}$ ومنه $\alpha_n < e^{\frac{1}{n}}$

* لدينا $1<\alpha_n<e^{\frac{1}{n}}$

و $\lim\limits_{+\infty} e^{\frac{1}{n}} = 1$

وبالتالي

$\lim\limits_{+\infty}\alpha_n = \boxed{1}$

* بين أن $\lim\limits_{+\infty} n(\alpha_n - 1) = 1$

$g_n(\alpha_n) = 0 \Leftrightarrow n = \dfrac{1}{\alpha_n \ln \alpha_n}$

ومنه $\lim\limits_{+\infty} n(\alpha_n - 1) = \lim\limits_{+\infty} \dfrac{1}{\alpha_n} \cdot \dfrac{\alpha_n - 1}{\ln \alpha_n}$

$\lim\limits_{+\infty} \alpha_n = 1 \Longrightarrow \lim\limits_{+\infty} \dfrac{\ln \alpha_n}{\alpha_n - 1} = 1$

و $\lim\limits_{+\infty} \dfrac{1}{\alpha_n} = 1$

إذن $\lim\limits_{+\infty} n(\alpha_n - 1) = \boxed{1}$

II

$$\begin{cases} f_n(x) = \dfrac{e^{nx}}{\ln x} \\ f_n(0) = 0 \end{cases}$$

1) نهايات f_n

$\lim\limits_{x \to 1} f_n(x) = \lim\limits_{x \to 1} \dfrac{e^{nx}}{\ln(x)}$

لدينا $\lim\limits_{x \to 1} e^{nx} = e$

و $\lim\limits_{x \to 1} \ln(x) = 0$

$x > 1 \Longrightarrow \ln(x) > 0$

$\Longrightarrow \lim\limits_{x \to 1^+} f_n(x) = +\infty$

$x < 1 \Longrightarrow \ln(x) < 0$

$\lim\limits_{x \to 1^-} f_n(x) = -\infty$

$\lim\limits_{+\infty} \dfrac{e^{nx}}{\ln(x)} = \lim\limits_{+\infty} \dfrac{e^{nx}}{nx} \cdot \dfrac{x}{\ln(x)}$

$\lim\limits_{+\infty} \dfrac{e^{nx}}{nx} = +\infty$

$\lim\limits_{+\infty} \dfrac{x}{\ln(x)} = +\infty$

ومنه $\lim\limits_{+\infty} f_n(x) = \boxed{+\infty}$

2) لدينا $\lim\limits_{0^+} f_n(x) = \lim\limits_{0^+} \dfrac{e^{nx}}{\ln(x)}$

$\lim\limits_{0^+} \dfrac{e^{nx}}{\ln x} = 0 \Leftarrow \begin{cases} \lim\limits_{0^+} e^{nx} = e^0 = 1 \\ \lim\limits_{0^+} \ln x = -\infty \end{cases}$

$\Leftarrow \lim\limits_{0^+} f_n(x) = f_n(0)$

f_n متصلة على يمين 0 .

ب) $\lim\limits_{0^+} \dfrac{f_n(x) - f_n(0)}{x - 0} = \lim\limits_{0^+} \dfrac{e^{nx}}{x \ln x}$

$\lim\limits_{0^+} \dfrac{e^{nx}}{x \ln x} = -\infty \Leftarrow \begin{cases} \lim\limits_{0^+} e^{nx} = 1 \\ \lim\limits_{0^+} x \ln x = 0^- \end{cases}$

f_n غير قابلة للاشتقاق على يمين 0 و C_n يقبل على يمين 0 نصف مماس عمودي.

3) $f_n'(x) = \dfrac{n \cdot e^{nx} \ln(x) - \dfrac{e^{nx}}{x}}{(\ln(x))^2}$

$f_n'(x) = \dfrac{\cancel{x} \cdot e^{nx}\left(n \ln(x) - \dfrac{1}{x}\right)}{\cancel{x} (\ln(x))^2}$

$f_n'(x) = \dfrac{e^{n \cdot x}}{(\ln(x))^2} \cdot g_n(x)$

①

$$a_0 = \int_0^{\pi/2} 1\, dt = [t]_0^{\frac{\pi}{2}} = \frac{\pi}{2}$$

$$b_0 = \int_0^{\pi/2} t^2 dt = \left[\frac{1}{3} t^3\right]_0^{\frac{\pi}{2}} = \frac{\pi^3}{24}$$

②

$$a_{n+2} = \int_0^{\pi/2} \cos^{n+2}(t)\, dt$$

$$= \int_0^{\pi/2} \cos t \cdot \cos^{n+1} t\, dt$$

نضع :

$$\begin{cases} u = \cos^{n+1} t \\ v' = \cos t \end{cases} \Rightarrow \begin{array}{l} u' = -(n+1)\cos^n t \sin t \\ v = \sin t \end{array}$$

$$a_{n+2} = \left[\sin t \cos^{n+1} t\right]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} (n+1)\cos^n t \cdot \sin^2 t\, dt$$

$$= (1\times 0 - 1\times 0) + (n+1)\int_0^{\pi/2} \cos^n t\,(1 - \cos^2 t)\, dt$$

$$a_{n+2} = (n+1)\int_0^{\frac{\pi}{2}} \cos^n t - (n+1)\int_0^{\pi/2} \cos^{n+2} t\, dt$$

$$(n+2)\, a_{n+2} = (n+1)\, a_n$$

$$\boxed{a_{n+2} = \frac{n+1}{n+2}\, a_n}$$

③

ب) استنتج $0 \leq b_n \leq \frac{\pi^2}{4}\left(a_{2n} - a_{2n+2}\right)$

لدينا $0 \leq t \leq \frac{\pi}{2} \sin t$

$$0 \leq t^2 \leq \frac{\pi^2}{4} \sin^2 t$$

و $\forall t \in \left[0, \frac{\pi}{2}\right] ;\ \cos^{2n} t \geq 0$

$$f_n'(x) = \frac{e^{nx}}{(\ln(x))^2} \cdot g_n(x)$$

لدينا $\ln(x)^2 > 0$ و $e^{nx} > 0$

إشارة $f_n'(x)$ هي إشارة $g_n(x)$

x	0		1		α_n		$+\infty$
$f_n'(x)$		$-$	||	$-$	0	$+$	
$f_n(x)$	0	$\searrow$	$-\infty$ || $+\infty$	$\searrow$	$f_n(\alpha_n)$	$\nearrow$	$+\infty$

4)

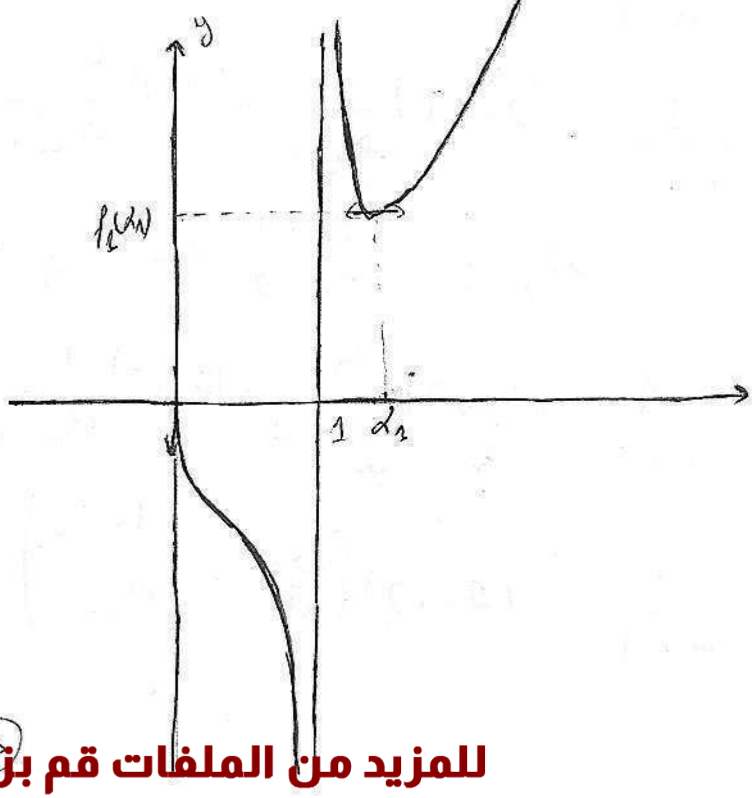
$$\lim_{+\infty} \frac{f_n(x)}{x} = \lim_{+\infty} \frac{e^{nx}}{x \cdot \ln(x)}$$

$$= \lim_{+\infty} n^2 \cdot \frac{e^{nx}}{(n\cdot x)^2} \times \frac{x}{\ln x}$$

$$\lim_{+\infty} \frac{e^{nx}}{(nx)^2} = +\infty \quad \text{و} \quad \lim_{+\infty} \frac{x}{\ln x} = +\infty$$

$$\Leftarrow \quad \lim_{+\infty} \frac{f_n(x)}{x} = +\infty$$

C_n يقبل فرع شلجميا في اتجاه محور الأراتيب



$a_{2n+2} = 2\int_0^{\pi/2} t.\sin t.\cos^{2n+1} t\,dt$ (3

$J_n = \int_0^{\pi/2} t.\sin t.\cos^{2n+1}(t)\,dt$ نضع

$u = \frac{1}{2}t^2 \Leftarrow \begin{cases} u' = t \\ v = \sin t.\cos^{2n+1} t \end{cases}$

$v'(t) = \cos^{2n+2} t - (2n+1)\cos^{2n} t \cdot \sin^2 t$

$v'(t) = (2n+2)\cos^{2n+2} - (2n+1)\cos^{2n}(t)$

$J_n = \left[\frac{1}{2}t^2 \sin(t)\cos^{2n+1}(t)\right]_0^{\frac{\pi}{2}}$

$-\frac{1}{2}\int_0^{\pi/2}\left((2n+2)t^2\cos^{2n+2} t - (2n+1)t^2\cos^{2n} t\right)dt$

$= \frac{1}{2}\int_0^{\pi/2}(2n+1)t^2.\cos^{2n} t\,dt - \frac{1}{2}\int_0^{\pi/2}(2n+2)t^2\cos^{2n+2} t\,dt$

$J_n = \frac{1}{2}(2n+1)b_n - \frac{1}{2}(2n+2)b_{n+1}$

ومنه

$\frac{a_{2n+2}}{n+1} = (2n+1)b_n - (2n+2)b_{n+1}$

استنتاج ،

$\frac{a_{2n+2}}{n+1} = (2n+1)b_n - (2n+2)b_{n+1}$

$\frac{1}{n+1} = (2n+1)\frac{b_n}{a_{2n+2}} - (2n+2)\frac{b_{n+1}}{a_{2n+2}}$

$a_{2n+2} = \frac{2n+1}{2n+2}a_n$ و

$\frac{1}{n+1} = (\cancel{2n+1})\frac{b_n}{\frac{\cancel{2n+1}}{2n+2}a_n} - (2n+2)\frac{b_{n+1}}{a_{2n+2}}$

$\frac{1}{n+1} = (2n+2)\left(\frac{b_n}{a_{2n}} - \frac{b_{n+1}}{a_{2n+2}}\right)$

$0 \leq t^2\cos^{2n} t \leq \frac{\pi^2}{4}\cos^{2n} t \cdot \sin^2 t$

$0 \leq t^2\cos^{2n} t \leq \frac{\pi^2}{4}\left(\cos^{2n} t - \cos^{2n+2} t\right)$

$0 \leq \int_0^{\frac{\pi}{2}} t^2\cos^{2n} t\,dt \leq \frac{\pi^2}{4}\int_0^{\pi/2}\left(\cos^{2n} t - \cos^{2n+2} t\right)dt$

إذن $\boxed{0 \leq b_n \leq \frac{\pi^2}{4}(a_{2n} - a_{2n+2})}$

$a_{2n} > 0$،

$0 \leq \frac{b_n}{a_{2n}} \leq \frac{\pi^2}{4}\left(1 - \frac{a_{2n+2}}{a_{2n}}\right)$

$a_{2n+2} = \frac{2n+1}{2n+2}a_{2n}$

$0 \leq \frac{b_n}{a_{2n}} \leq \frac{\pi^2}{4} \cdot \frac{1}{2n+2}$

بما أن $\lim_{+\infty} \frac{\pi^2}{4} \cdot \frac{1}{2n+2} = 0$

إذن $\boxed{\lim_{+\infty} \frac{b_n}{a_{2n}} = 0}$

$a_{2n+2} = (2n+2)\int_0^{\pi/2} t\sin t\cos^{2n+1} t\,dt$ (4

لدينا

$a_{2n+2} = \int_0^{\pi/2}\cos^{2n+2}(t)\,dt$

$u(t) = t$

$v'(t) = -(2n+2)\sin t.\cos^{2n+1} t \Leftarrow \begin{cases} u'(t) = 1 \\ v(t) = \cos^{2n+2} t \end{cases}$

$a_{2n+2} = \left[t.\cos^{2n+2}\right]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}}(2n+2)t\sin(t)\cos^{2n+1}$

$= (2n+2)\int_0^{\pi/2} t.\sin(t).\cos^{2n+1} t\,dt$

و عليه $\frac{1}{(n+1)^2} = 2\left(\frac{b_n}{a_{2n}} - \frac{b_{n+1}}{a_{2n+2}}\right)$

* $U_n = \sum_{1}^{n} \frac{1}{k^2}$

لدينا $\frac{1}{k^2} = 2\left(\frac{b_{k-1}}{a_{2(k-1)}} - \frac{b_k}{a_{2k}}\right)$

$$\sum_{1}^{n} \frac{1}{k^2} = 2\sum_{1}^{n}\left(\frac{b_{k-1}}{a_{2(k-1)}} - \frac{b_k}{a_{2k}}\right)$$

$$= 2\left(\frac{b_0}{a_0} - \frac{b_n}{a_{2n}}\right)$$

لدينا $\lim_{+\infty} \frac{b_n}{a_{2n}} = 0$

$$\frac{b_0}{a_0} = \frac{\frac{\pi^3}{24}}{\frac{\pi}{2}} = \frac{\pi^2}{12}$$

و منه $\lim_{+\infty} U_n = \boxed{\frac{\pi^2}{6}}$

$$V_n = \sum_{0}^{n} \frac{1}{(2k+1)^2}$$

$$U_{2n+1} = \sum_{1}^{2n+1} \frac{1}{k^2}$$

$$= \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots + \frac{1}{(2n+1)^2}$$

$$= \left(\frac{1}{1^2} + \frac{1}{3^2} + \dots + \frac{1}{(2n+1)^2}\right) + \left(\frac{1}{2^2} + \frac{1}{4^2} + \dots + \frac{1}{(2n)^2}\right)$$

$$= \sum_{0}^{n} \frac{1}{(2k+1)^2} + \sum_{1}^{n} \frac{1}{(2k)^2}$$

$$U_{2n+1} = V_n + \frac{1}{4} U_n$$

$$V_n = U_{2n+1} - \frac{1}{4} U_n$$

$$\Longrightarrow \lim_{+\infty} V_n = \frac{\pi^2}{6} - \frac{\pi^2}{24} = \boxed{\frac{\pi^2}{8}}$$

من إنجاز: عمارة محمد يزيد

3) نضع $\varphi(t) = t - \frac{\pi}{2}\sin t$

و $t \in [0, \frac{\pi}{2}]$

لدينا $\varphi'(t) = 1 - \frac{\pi}{2}\cos t$

و $\forall t \in [0, \frac{\pi}{2}]\ \ \varphi''(t) = \frac{\pi}{2}\sin t \geq 0$

⇐

t	0		α		$\frac{\pi}{2}$
متصلة φ'	$1-\frac{\pi}{2}<0$	↗	0	↗	1

⇐ $\exists!\alpha \in]0, \frac{\pi}{2}[\ \ \varphi'(\alpha) = 0$

و منه

t	0		α		$\frac{\pi}{2}$
$\varphi'(t)$		$-$	0	$+$	
φ	0	↘	$\varphi(\alpha)$	↗	0

وعليه، $\forall t \in [0, \frac{\pi}{2}]\ \ \varphi(t) \leq 0$

وبالتالي $\forall t \in [0, \frac{\pi}{2}] :\ t \leq \frac{\pi}{2}\sin t$