

تمارين حو النهايات

تمارين و حلولها

حدد النهايات التالية

$$\begin{aligned} & \lim_{x \rightarrow -\infty} \frac{-3x^6 + 2x^2 + 1}{x^3 + 3x - 1} \text{ و } \lim_{x \rightarrow +\infty} \frac{2x^5 + 3x^3 + x}{-5x^5 + 2x - 1} \text{ و } \lim_{x \rightarrow -\infty} -2x^4 + 3x^3 + x - 1 \text{ و } \lim_{x \rightarrow 2} 3 + x - 3x^2 \\ & \lim_{x \rightarrow 2^+} \frac{-2x + 1}{x^2 - x - 2} \text{ و } \lim_{x \rightarrow 1^+} \frac{-2x + 1}{x^2 - 3x + 2} \text{ و } \lim_{x \rightarrow 1} \frac{2x^2 - 5x + 3}{x^2 + 2x - 3} \text{ و } \lim_{x \rightarrow +\infty} \frac{7x^4 + x^2 + 1}{-x^8 - 2x - 1} \text{ و } \\ & \lim_{x \rightarrow -\infty} \sqrt{x^2 + x} + x \text{ و } \lim_{x \rightarrow +\infty} \sqrt{x^2 + x} - x \text{ و } \lim_{x \rightarrow 1} \frac{\sqrt{2x-1} - 1}{x-1} \text{ و } \lim_{x \rightarrow 2^-} \frac{-2x + 1}{x^2 - x - 2} \text{ و } \\ & \lim_{x \rightarrow 0} \frac{\sqrt{x+1} - 1}{\tan x} \text{ و } \lim_{x \rightarrow \frac{\pi}{2}} \frac{1}{\tan^2 x + 2} \text{ و } \lim_{x \rightarrow 0} \frac{\sin 5x}{\sin x} \text{ و } \lim_{x \rightarrow +\infty} \frac{\sqrt{x + \sqrt{x}} + \sqrt{x}}{\sqrt{x+1}} \text{ و } \lim_{x \rightarrow +\infty} \frac{x}{x - 2\sqrt{x+1}} \\ & \lim_{x \rightarrow \frac{\pi}{2}^+} \frac{\cos x}{1 - \sin x} \text{ و } \lim_{x \rightarrow 0} \frac{\cos 3x - \cos x}{\sin 3x + \sin x} \text{ و } \end{aligned}$$

الجواب

نحدد النهايات

$$\lim_{x \rightarrow -\infty} -2x^4 + 3x^3 + x - 1 = \lim_{x \rightarrow -\infty} -2x^4 = -\infty \quad \lim_{x \rightarrow 2} 3 + x - 3x^2 = -7 \quad *$$

$$\lim_{x \rightarrow +\infty} \frac{2x^5 + 3x^3 + x}{-5x^5 + 2x - 1} = \lim_{x \rightarrow +\infty} \frac{2x^5}{-5x^5} = -\frac{2}{5} \quad *$$

$$\lim_{x \rightarrow -\infty} \frac{-3x^6 + 2x^2 + 1}{x^3 + 3x - 1} = \lim_{x \rightarrow -\infty} \frac{-3x^6}{x^3} = \lim_{x \rightarrow -\infty} -3x^3 = +\infty \quad *$$

$$\lim_{x \rightarrow +\infty} \frac{7x^4 + x^2 + 1}{-x^8 - 2x - 1} = \lim_{x \rightarrow +\infty} \frac{7x^4}{-x^8} = \lim_{x \rightarrow +\infty} \frac{-7}{x^4} = 0 \quad *$$

$$\lim_{x \rightarrow 1} \frac{2x^2 - 5x + 3}{x^2 + 2x - 3} = \lim_{x \rightarrow 1} \frac{(x-1)(2x-3)}{(x-1)(x+3)} = \lim_{x \rightarrow 1} \frac{2x-3}{x+3} = -\frac{1}{4} \quad *$$

$$\lim_{x \rightarrow 1^+} \frac{-2x+1}{x^2-3x+2} \quad \text{نحدد} *$$

x	$-\infty$	1	2	$+\infty$	
$x^2 - 3x + 2$	+	0	-	0	+

$$\lim_{x \rightarrow 1^+} \frac{-2x+1}{x^2-3x+2} = +\infty \quad \text{ومنه} \quad \lim_{x \rightarrow 1^+} -2x+1 = -1 \quad \lim_{x \rightarrow 1^+} x^2 - 3x + 2 = 0^-$$

* لدينا $\lim_{x \rightarrow 2} -2x + 1 = -3$ و $\lim_{x \rightarrow 2} x^2 - x - 2 = 0$

x	$-\infty$	-1	2	$+\infty$	
$x^2 - x - 2$	$+$	0	$-$	0	$+$

ومنه $\lim_{x \rightarrow 2^-} x^2 - x - 2 = 0^-$ و $\lim_{x \rightarrow 2^+} x^2 - x - 2 = 0^+$

إذن $\lim_{x \rightarrow 2^+} \frac{-2x+1}{x^2-x-2} = -\infty$; $\lim_{x \rightarrow 2^-} \frac{-2x+1}{x^2-x-2} = +\infty$

* $\lim_{x \rightarrow 1} \frac{\sqrt{2x-1}-1}{x-1} = \lim_{x \rightarrow 1} \frac{2x-2}{(x-1)(\sqrt{2x-1}+1)} = \lim_{x \rightarrow 1} \frac{2}{(\sqrt{2x-1}+1)} = 1$

* لدينا $\lim_{x \rightarrow +\infty} \sqrt{x^2+x} - x = \lim_{x \rightarrow +\infty} \frac{x}{\sqrt{x^2+x}+x} = \lim_{x \rightarrow +\infty} \frac{x}{\sqrt{x^2\left(1+\frac{1}{x}\right)}+x} = \lim_{x \rightarrow +\infty} \frac{x}{|x|\sqrt{1+\frac{1}{x}}+x}$

و حيث x تتؤول إلى $+\infty$ فإن x موجبة ومنه $|x| = x$

ومنه $\lim_{x \rightarrow +\infty} \sqrt{x^2+x} - x = \lim_{x \rightarrow +\infty} \frac{x}{x\left(\sqrt{1+\frac{1}{x}}+1\right)} = \lim_{x \rightarrow +\infty} \frac{1}{\sqrt{1+\frac{1}{x}}+1} = \frac{1}{2}$

* $\lim_{x \rightarrow -\infty} \sqrt{x^2+x} + x = \lim_{x \rightarrow -\infty} \frac{x}{\sqrt{x^2+x}-x} = \lim_{x \rightarrow -\infty} \frac{x}{\sqrt{x^2\left(1+\frac{1}{x}\right)}-x} = \lim_{x \rightarrow -\infty} \frac{x}{|x|\sqrt{1+\frac{1}{x}}-x}$

و حيث x تتؤول إلى $-\infty$ فإن x سالبة ومنه $|x| = -x$

إذن $\lim_{x \rightarrow -\infty} \sqrt{x^2+x} + x = \lim_{x \rightarrow -\infty} \frac{x}{-x\sqrt{1+\frac{1}{x}}-x} = \lim_{x \rightarrow -\infty} \frac{1}{-\sqrt{1+\frac{1}{x}}-1} = -\frac{1}{2}$

* $\lim_{x \rightarrow +\infty} \frac{x}{x-2\sqrt{x+1}} = \lim_{x \rightarrow +\infty} \frac{x}{x-2\sqrt{x^2\left(\frac{1}{x}+\frac{1}{x^2}\right)}} = \lim_{x \rightarrow +\infty} \frac{x}{x-2|x|\sqrt{\frac{1}{x}+\frac{1}{x^2}}}$

و حيث x تتؤول إلى $+\infty$ فإن x موجبة ومنه $|x| = x$

$\lim_{x \rightarrow +\infty} \frac{x}{x-2\sqrt{x+1}} = \lim_{x \rightarrow +\infty} \frac{x}{x\left(1-2\sqrt{\frac{1}{x}+\frac{1}{x^2}}\right)} = \lim_{x \rightarrow +\infty} \frac{1}{1-2\sqrt{\frac{1}{x}+\frac{1}{x^2}}} = 1$

* $\lim_{x \rightarrow +\infty} \frac{\sqrt{x}+\sqrt{x}+\sqrt{x}}{\sqrt{x+1}} = \lim_{x \rightarrow +\infty} \frac{\sqrt{x\left(1+\frac{1}{\sqrt{x}}\right)}+\sqrt{x}}{\sqrt{x+1}} = \lim_{x \rightarrow +\infty} \frac{\sqrt{x}\left(\sqrt{1+\frac{1}{\sqrt{x}}}+1\right)}{\sqrt{x}\sqrt{1+\frac{1}{x}}} = \lim_{x \rightarrow +\infty} \frac{\sqrt{1+\frac{1}{\sqrt{x}}}+1}{\sqrt{1+\frac{1}{x}}} = 2$

$$\lim_{x \rightarrow 0} \frac{\sin 5x}{\sin x} = \lim_{x \rightarrow 0} \frac{x}{\sin x} \times \frac{\sin 5x}{5x} \times 5 \quad \text{لدينا} \quad *$$

$$\lim_{x \rightarrow 0} \frac{x}{\sin x} = 1 \quad \text{ومنه} \quad \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \quad \text{نعلم أن}$$

$$\lim_{x \rightarrow 0} \frac{\sin 5x}{\sin x} = 1 \times 1 \times 5 = 5 \quad \text{فان} \quad \lim_{x \rightarrow 0} \frac{\sin 5x}{5x} = 1 \quad \text{و حيث}$$

$$\lim_{x \rightarrow \frac{\pi}{2}} \frac{1}{\tan^2 x + 2} = 0 \quad \text{ومنه} \quad \lim_{x \rightarrow \frac{\pi}{2}} \tan^2 x = +\infty \quad \text{لدينا} \quad *$$

$$\lim_{x \rightarrow 0} \frac{\sqrt{x+1}-1}{\tan x} = \lim_{x \rightarrow 0} \frac{x}{\tan x} \times \frac{1}{\sqrt{x+1}+1} \quad *$$

$$\lim_{x \rightarrow 0} \frac{x}{\tan x} = 1 \quad \text{ومنه} \quad \lim_{x \rightarrow 0} \frac{\tan x}{x} = 1 \quad \text{نعلم أن}$$

$$\lim_{x \rightarrow 0} \frac{\sqrt{x+1}-1}{\tan x} = 1 \times \frac{1}{2} = \frac{1}{2} \quad \text{إذن} \quad \lim_{x \rightarrow 0} \sqrt{x+1}+1 = 2 \quad \text{لدينا}$$

$$\lim_{x \rightarrow 0} \frac{\cos 3x - \cos x}{\sin 3x + \sin x} = \lim_{x \rightarrow 0} \frac{-2 \sin 2x \cdot \sin x}{2 \sin 2x \cdot \cos x} = \lim_{x \rightarrow 0} -\tan x = 0 \quad *$$

$$\lim_{x \rightarrow \frac{\pi}{2}^+} \frac{\cos x}{1 - \sin x} = \lim_{x \rightarrow \frac{\pi}{2}^+} \frac{(1 + \sin x) \cos x}{1 - \sin^2 x} = \lim_{x \rightarrow \frac{\pi}{2}^+} \frac{1 + \sin x}{\cos x} = \quad \text{لدينا} \quad *$$

$$\forall x \in \left] -\pi; \frac{\pi}{2} \right[\quad \cos x < 0 \quad \text{لأن} \quad \lim_{x \rightarrow \frac{\pi}{2}^+} \cos x = 0^- \quad \text{و} \quad \lim_{x \rightarrow \frac{\pi}{2}^+} 1 + \sin x = 2 \quad \text{لدينا}$$

$$\lim_{x \rightarrow \frac{\pi}{2}^+} \frac{\cos x}{1 - \sin x} = \lim_{x \rightarrow \frac{\pi}{2}^+} \frac{1 + \sin x}{\cos x} = -\infty \quad \text{ومنه}$$

تمرين 2

$$\lim_{x \rightarrow 0} x \sin \frac{1}{x} = 0 \quad \text{بين أن}$$

الجواب

$$\forall x \in \mathbb{R}^* \quad \left| x \sin \frac{1}{x} \right| \leq |x| \quad \text{ومنه} \quad \forall x \in \mathbb{R}^* \quad -1 \leq \sin \frac{1}{x} \leq 1 \quad \text{لدينا}$$

$$\lim_{x \rightarrow 0} x \sin \frac{1}{x} = 0 \quad \text{فان} \quad \lim_{x \rightarrow 0} |x| = 0 \quad \text{و حيث أن}$$

تمرين 3

$$g(x) = \frac{\sqrt{2 - \cos x} - 1}{x^2} \quad \text{نعتبر } g \text{ دالتين عددية للمتغير حقيقي } x \text{ حيث}$$

$$\lim_{x \rightarrow 0} g(x) \quad \text{حدد -1}$$

$$\lim_{x \rightarrow +\infty} g(x) \quad \text{واستنتج} \quad \forall x \in \mathbb{R}^* \quad |g(x)| \leq \frac{1}{x^2} \quad \text{-3 بين أن}$$

الجواب

1- نحدد $\lim_{x \rightarrow 0} g(x)$

$$\lim_{x \rightarrow 0} g(x) = \lim_{x \rightarrow 0} \frac{\sqrt{2 - \cos x} - 1}{x^2} = \lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2} \times \frac{1}{\sqrt{2 - \cos x} + 1}$$

لدينا $\lim_{x \rightarrow 0} 2 - \cos x = 1$ ومنه $\lim_{x \rightarrow 0} \frac{1}{\sqrt{2 - \cos x} + 1} = \frac{1}{2}$

وحيث $\lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2} = \frac{1}{2}$ فان $\lim_{x \rightarrow 0} g(x) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

2- نبين أن $\forall x \in \mathbb{R}^* \quad |g(x)| \leq \frac{1}{x^2}$

$$\forall x \in \mathbb{R}^* \quad -1 \leq \cos x \leq 1 \Leftrightarrow 1 \leq 2 - \cos x \leq 3 \Leftrightarrow 0 \leq \sqrt{2 - \cos x} - 1 \leq \sqrt{3} - 1 < 1$$

$$\forall x \in \mathbb{R}^* \quad |g(x)| = \left| \frac{\sqrt{2 - \cos x} - 1}{x^2} \right| = \frac{\sqrt{2 - \cos x} - 1}{x^2}$$

لدينا $\forall x \in \mathbb{R}^* \quad 0 \leq \sqrt{2 - \cos x} - 1 < 1$ ومنه $\forall x \in \mathbb{R}^* \quad 0 \leq \frac{\sqrt{2 - \cos x} - 1}{x^2} < \frac{1}{x^2}$

$$\forall x \in \mathbb{R}^* \quad |g(x)| \leq \frac{1}{x^2} \quad \text{إذن}$$

نستنتج $\lim_{x \rightarrow +\infty} g(x)$

لدينا $\forall x \in \mathbb{R}^* \quad |g(x)| \leq \frac{1}{x^2}$ و $\lim_{x \rightarrow +\infty} \frac{1}{x^2} = 0$ و منه $\lim_{x \rightarrow +\infty} g(x) = 0$