

رياضيات النجاح	نهاية دالة عددية حلول مقترحة	السنة 1 بكالوريا علوم رياضية
تمرين 1 :		
$\lim_{x \rightarrow +\infty} (\sqrt{9x^2 + x + 1} - 2x) = \lim_{x \rightarrow +\infty} \left(\sqrt{x^2 \left(9 + \frac{1}{x} + \frac{1}{x^2} \right)} - 2x \right) = \lim_{x \rightarrow +\infty} \left(x \sqrt{9 + \frac{1}{x} + \frac{1}{x^2}} - 2x \right)$ $\lim_{x \rightarrow +\infty} (\sqrt{9x^2 + x + 1} - 2x) = \lim_{x \rightarrow +\infty} x \left(\sqrt{9 + \frac{1}{x} + \frac{1}{x^2}} - 2 \right) = +\infty$ (لأن : $\lim_{x \rightarrow +\infty} \sqrt{9 + \frac{1}{x} + \frac{1}{x^2}} - 2 = 3 - 2 = 1$ و $\lim_{x \rightarrow +\infty} x = +\infty$)		
$\lim_{x \rightarrow -\infty} (\sqrt{9x^2 + x + 1} + 3x) = \lim_{x \rightarrow -\infty} \frac{9x^2 + x + 1 - 9x^2}{\sqrt{9x^2 + x + 1} - 3x} = \lim_{x \rightarrow -\infty} \frac{x + 1}{\sqrt{x^2 \left(9 + \frac{1}{x} + \frac{1}{x^2} \right)} - 3x}$ $\lim_{x \rightarrow -\infty} (\sqrt{9x^2 + x + 1} + 3x) = \lim_{x \rightarrow -\infty} \frac{x \left(1 + \frac{1}{x} \right)}{-x \sqrt{9 + \frac{1}{x} + \frac{1}{x^2}} - 3x} = \lim_{x \rightarrow -\infty} \frac{1 + \frac{1}{x}}{-\sqrt{9 + \frac{1}{x} + \frac{1}{x^2}} - 3} = \frac{1}{-3 - 3} = -\frac{1}{6}$		
انتبه أنه إذا كان : $x < 0$ فإن : $\sqrt{x^2} = -x = -x$ الطريقة السابقة لن تجدي في هذه النهاية لأننا سنحصل على شكل غير محدد المثالان متشابهان لكن الطريقة تختلف و السبب المعاملات الموجودة في كل نهاية، لذلك نأمل أن تدرك متى نستعمل المرافق ومتى نعمل بأكبر أس.		
$\lim_{x \rightarrow +\infty} (x^2 - \sqrt{x + 2}) = \lim_{x \rightarrow +\infty} \left(x^2 - \sqrt{x^2 \left(\frac{1}{x} + \frac{2}{x} \right)} \right) = \lim_{x \rightarrow +\infty} \left(x^2 - x \sqrt{\frac{1}{x} + \frac{2}{x}} \right) = \lim_{x \rightarrow +\infty} x \left(x - \sqrt{\frac{1}{x} + \frac{2}{x}} \right) = +\infty$ (لأن : $\lim_{x \rightarrow +\infty} x = +\infty$ و $\lim_{x \rightarrow +\infty} x - \sqrt{\frac{1}{x} + \frac{2}{x}} = +\infty$)		
$\lim_{x \rightarrow +\infty} \sqrt{1 + 2x^3} - \sqrt{x^3 + x + 1} = \lim_{x \rightarrow +\infty} \sqrt{x^3} \sqrt{\frac{1}{x^3} + 2} - \sqrt{x^3} \sqrt{1 + \frac{1}{x^2} + \frac{1}{x^3}}$ $\lim_{x \rightarrow +\infty} \sqrt{1 + 2x^3} - \sqrt{x^3 + x + 1} = \lim_{x \rightarrow +\infty} \sqrt{x^3} \left(\sqrt{\frac{1}{x^3} + 2} - \sqrt{1 + \frac{1}{x^2} + \frac{1}{x^3}} \right) = +\infty$ (لأن : $\lim_{x \rightarrow +\infty} \sqrt{x^3} = +\infty$ و $\lim_{x \rightarrow +\infty} \sqrt{\frac{1}{x^3} + 2} - \sqrt{1 + \frac{1}{x^2} + \frac{1}{x^3}} = \sqrt{2} - 1 > 0$)		
■ إذا كان $m > 0$ فإن : $\lim_{x \rightarrow -\infty} (\sqrt{4x^2 + 3x + 7} - mx) = +\infty$ ($\lim_{x \rightarrow -\infty} -mx = +\infty$) ■ إذا كان $m = 0$ فإن : $\lim_{x \rightarrow -\infty} (\sqrt{4x^2 + 3x + 7} - mx) = \lim_{x \rightarrow -\infty} (\sqrt{4x^2 + 3x + 7}) = +\infty$ ■ إذا كان $-2 < m < 0$ فإن : $\lim_{x \rightarrow -\infty} (\sqrt{4x^2 + 3x + 7} - mx) = \lim_{x \rightarrow -\infty} \left(\sqrt{x^2 \left(4 + \frac{3}{x} + \frac{7}{x^2} \right)} - mx \right) = \lim_{x \rightarrow -\infty} \left(-x \sqrt{4 + \frac{3}{x} + \frac{7}{x^2}} - mx \right)$ $\lim_{x \rightarrow -\infty} (\sqrt{4x^2 + 3x + 7} - mx) = \lim_{x \rightarrow -\infty} -x \left(\sqrt{4 + \frac{3}{x} + \frac{7}{x^2}} + m \right) = +\infty \quad (+\infty \times (2 + m) > 0)$ ■ إذا كان $m = -2$ فإن :		

$$\lim_{x \rightarrow -\infty} (\sqrt{4x^2 + 3x + 7} - mx) = \lim_{x \rightarrow -\infty} \sqrt{4x^2 + 3x + 7} + 2x = \lim_{x \rightarrow -\infty} \frac{3x + 7}{\sqrt{4x^2 + 3x + 7} - 2x}$$

$$\lim_{x \rightarrow -\infty} (\sqrt{4x^2 + 3x + 7} - mx) = \lim_{x \rightarrow -\infty} \frac{x \left(3 + \frac{7}{x} \right)}{-x \sqrt{4 + \frac{3}{x} + \frac{7}{x^2}} - 2x} = \lim_{x \rightarrow -\infty} \frac{3 + \frac{7}{x}}{-\sqrt{4 + \frac{3}{x} + \frac{7}{x^2}} - 2} = \frac{-3}{4}$$

إذا كان $m < -2$ فإن :

$$\lim_{x \rightarrow -\infty} (\sqrt{4x^2 + 3x + 7} - mx) = \lim_{x \rightarrow -\infty} -x \left(\sqrt{4 + \frac{3}{x} + \frac{7}{x^2}} + m \right) = -\infty \quad (+\infty \times (2 + m) < 0)$$

سؤال يوضح بالتفاصيل متى نستعمل طريقة النهاية الأولى و متى نستعمل المرافق

$$\lim_{x \rightarrow +\infty} \left(\frac{x - \sqrt{x^2 + x + 1}}{x^2 - \sqrt{x^4 - 1}} \right) = \lim_{x \rightarrow +\infty} \frac{(x^2 - (x^2 + x + 1)) \left(x^2 + \sqrt{x^4 - 1} \right)}{(x^4 - (x^4 - 1)) \left(x + \sqrt{x^2 + x + 1} \right)} = \lim_{x \rightarrow +\infty} \frac{-(x+1) \left(x^2 + x^2 \sqrt{1 - \frac{1}{x^4}} \right)}{\left(x + x \sqrt{1 + \frac{1}{x} + \frac{1}{x^2}} \right)}$$

$$\lim_{x \rightarrow +\infty} \left(\frac{x - \sqrt{x^2 + x + 1}}{x^2 - \sqrt{x^4 - 1}} \right) = \lim_{x \rightarrow +\infty} \frac{-x(x+1) \left(1 + \sqrt{1 - \frac{1}{x^4}} \right)}{1 + \sqrt{1 + \frac{1}{x} + \frac{1}{x^2}}} = -\infty$$

$$\lim_{x \rightarrow -\infty} \left(\frac{x - \sqrt{x^2 + x + 1}}{x^2 - \sqrt{x^4 - 1}} \right) = \lim_{x \rightarrow -\infty} \frac{(x - \sqrt{x^2 + x + 1}) \left(x^2 + \sqrt{x^4 - 1} \right)}{(x^4 - (x^4 - 1))} = \lim_{x \rightarrow -\infty} (x - \sqrt{x^2 + x + 1}) \left(x^2 + x^2 \sqrt{1 - \frac{1}{x^4}} \right)$$

$$\lim_{x \rightarrow -\infty} \left(\frac{x - \sqrt{x^2 + x + 1}}{x^2 - \sqrt{x^4 - 1}} \right) = \lim_{x \rightarrow -\infty} x^2 (x - \sqrt{x^2 + x + 1}) \left(1 + \sqrt{1 - \frac{1}{x^4}} \right) = -\infty$$

(لأن : $\lim_{x \rightarrow -\infty} 1 + \sqrt{1 - \frac{1}{x^4}} = 1$ و $\lim_{x \rightarrow -\infty} x^2 = +\infty$ و $\lim_{x \rightarrow -\infty} x - \sqrt{x^2 + x + 1} = -\infty$ $(-\infty + -\infty)$)

$$\lim_{x \rightarrow -\infty} \left(\frac{x+1 - \sqrt{1-x}}{x^2 - \sqrt{x^2+2}} \right) = \lim_{x \rightarrow -\infty} \frac{x+1 - \sqrt{x^2 \left(\frac{1}{x^2} - \frac{1}{x} \right)}}{x^2 - \sqrt{x^2 \left(1 + \frac{2}{x^2} \right)}} = \lim_{x \rightarrow -\infty} \frac{x+1 + x \sqrt{\frac{1}{x^2} - \frac{1}{x}}}{x^2 + x \sqrt{1 + \frac{2}{x^2}}}$$

$$= \lim_{x \rightarrow -\infty} \frac{x \left(1 + \frac{1}{x} + \sqrt{\frac{1}{x^2} - \frac{1}{x}} \right)}{x \left(x + \sqrt{1 + \frac{2}{x^2}} \right)} = \lim_{x \rightarrow -\infty} \frac{1 + \frac{1}{x} + \sqrt{\frac{1}{x^2} - \frac{1}{x}}}{x + \sqrt{1 + \frac{2}{x^2}}} = 0$$

(لأن : $\lim_{x \rightarrow +\infty} 1 + \frac{1}{x} + \sqrt{\frac{1}{x^2} - \frac{1}{x}} = 1$ و $\lim_{x \rightarrow +\infty} x + \sqrt{1 + \frac{2}{x^2}} = -\infty$)

تمرين 2 :

$$\lim_{x \rightarrow -1} \left(\frac{\sqrt{1-3x}-2}{x+1} \right) = \lim_{x \rightarrow -1} \frac{1-3x-4}{(x+1)(\sqrt{1-3x}+2)} = \lim_{x \rightarrow -1} \frac{-3}{\sqrt{1-3x}+2} = \frac{-3}{4}$$

$$\lim_{x \rightarrow 2^+} \frac{\sqrt{x^2-4}}{x-2} = \lim_{x \rightarrow 2^+} \sqrt{\frac{(x-2)(x+2)}{(x-2)^2}} = \lim_{x \rightarrow 2^+} \sqrt{\frac{x+2}{x-2}} = +\infty$$

$$\lim_{x \rightarrow -2^-} \frac{\sqrt{x^2 - 4}}{x + 2} = \lim_{x \rightarrow -2^-} -\frac{\sqrt{x^2 - 4}}{-(x + 2)} = \lim_{x \rightarrow 2^+} -\sqrt{\frac{(x - 2)(x + 2)}{(x + 2)^2}} = \lim_{x \rightarrow 2^+} -\sqrt{\frac{x - 2}{x + 2}} = -\infty$$

$$\lim_{x \rightarrow 1} \left(\frac{x^2 + x - 2}{x^3 + 4x^2 - 8x + 3} \right) = \lim_{x \rightarrow 1} \frac{(x - 1)(x + 2)}{(x - 1)(x^2 + 5x - 3)} = 1$$

استعملنا القسمة الإقليدية على $(x - 1)$ للتعميل في البسط و المقام

$$\begin{aligned} \lim_{x \rightarrow -1} \left(\frac{\sqrt{2 + x} + \sqrt{3 - x} - 3}{x + 1} \right) &= \lim_{x \rightarrow -1} \left(\frac{\sqrt{2 + x} - 1}{x + 1} + \frac{\sqrt{3 - x} - 2}{x + 1} \right) \\ &= \lim_{x \rightarrow -1} \left(\frac{x + 1}{(x + 1)(\sqrt{2 + x} + 1)} + \frac{-1 - x}{(x + 1)(\sqrt{3 - x} + 2)} \right) \\ &= \lim_{x \rightarrow -1} \left(\frac{1}{\sqrt{2 + x} + 1} + \frac{-1}{\sqrt{3 - x} + 2} \right) \end{aligned}$$

$$\lim_{x \rightarrow -1} \left(\frac{\sqrt{2 + x} + \sqrt{3 - x} - 3}{x + 1} \right) = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$

$$\lim_{x \rightarrow 0^+} \left(\frac{x + \sqrt{x}}{\sqrt{x^2 + x} - x} \right) = \lim_{x \rightarrow 0^+} \left(\frac{\sqrt{x^2 + x}}{\sqrt{x(x + 1)} - x} \right) = \lim_{x \rightarrow 0^+} \frac{\sqrt{x}(\sqrt{x + 1})}{\sqrt{x}\sqrt{x + 1} - \sqrt{x}^2} = \lim_{x \rightarrow 0^+} \frac{\sqrt{x}(\sqrt{x + 1})}{\sqrt{x}(\sqrt{x + 1} - \sqrt{x})}$$

$$\lim_{x \rightarrow 0^+} \left(\frac{x + \sqrt{x}}{\sqrt{x^2 + x} - x} \right) = \lim_{x \rightarrow 0^+} \frac{\sqrt{x} + 1}{\sqrt{x + 1} - \sqrt{x}} = 1$$

تمرين 3 :

$$\lim_{x \rightarrow 0} \left(\frac{\sqrt{3} \cos x - \sin^2 x - \sqrt{3}}{x^2} \right) = \lim_{x \rightarrow 0} \left(-\sqrt{3} \left(\frac{1 - \cos(x)}{x^2} \right) - \left(\frac{\sin(x)}{x} \right)^2 \right) = \frac{-\sqrt{3}}{2} - 1$$

$$\lim_{x \rightarrow 0} \left(\frac{1}{\sqrt{3} \cos x - \sin^2 x - \sqrt{3}} \right) = \lim_{x \rightarrow 0} \left(\frac{1}{\sqrt{3}(\cos x - 1) - \sin^2 x} \right) = \lim_{x \rightarrow 0} \left(\frac{-1}{\sqrt{3}(1 - \cos(x)) + \sin^2 x} \right)$$

$$\lim_{x \rightarrow 0} \left(\frac{1}{\sqrt{3} \cos x - \sin^2 x - \sqrt{3}} \right) = \lim_{x \rightarrow 0} \left(\frac{-\frac{1}{x^2}}{\sqrt{3} \left(\frac{1 - \cos(x)}{x^2} \right) + \left(\frac{\sin x}{x} \right)^2} \right) = -\infty$$

للتذكير : $\lim_{x \rightarrow 0} \frac{1 - \cos x}{x} = 1$ و $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$

$$\lim_{x \rightarrow \frac{p}{4}} \left(\frac{\tan x - 1}{2 \cos x - \sqrt{2}} \right) = \lim_{x \rightarrow \frac{p}{4}} \left(\frac{\tan x - \tan\left(\frac{p}{4}\right)}{2 \cos x - \sqrt{2}} \right) = \lim_{x \rightarrow \frac{p}{4}} \left(\frac{\left(1 + \tan(x) \tan\left(\frac{p}{4}\right)\right) \tan\left(x - \frac{p}{4}\right)}{2 \cos x - \sqrt{2}} \right)$$

لدينا :

الآن نضع $t = x - \frac{p}{4}$ فنجد :

$$\begin{aligned} \lim_{x \rightarrow \frac{p}{4}} \left(\frac{\tan x - 1}{2 \cos x - \sqrt{2}} \right) &= \lim_{t \rightarrow 0} \left(\frac{\left(1 + \tan \left(t + \frac{p}{4} \right) \right) \tan(t)}{2 \cos \left(t + \frac{p}{4} \right) - \sqrt{2}} \right) = \lim_{t \rightarrow 0} \left(\frac{\left(1 + \tan \left(t + \frac{p}{4} \right) \right) \tan(t)}{2 \left(\cos(t) \frac{\sqrt{2}}{2} - \sin(t) \frac{\sqrt{2}}{2} \right) - \sqrt{2}} \right) \\ &= \lim_{t \rightarrow 0} \left(\frac{\left(1 + \tan \left(t + \frac{p}{4} \right) \right) \tan(t)}{-\sqrt{2}(1 - \cos(t)) - \sqrt{2} \sin(t)} \right) = \lim_{t \rightarrow 0} \left(\frac{\left(1 + \tan \left(t + \frac{p}{4} \right) \right) \frac{\tan(t)}{t}}{-\sqrt{2} t \frac{(1 - \cos(t))}{t^2} - \sqrt{2} \frac{\sin(t)}{t}} \right) \\ \lim_{x \rightarrow \frac{p}{4}} \left(\frac{\tan x - 1}{2 \cos x - \sqrt{2}} \right) &= \frac{(1+1) \times 1}{-\sqrt{2} \times 0 \times \frac{1}{2} - \sqrt{2} \times 1} = -\sqrt{2} \end{aligned}$$

👉 لاحظ أن استعمال الخاصية $\tan(a-b) = \frac{\tan(a) - \tan(b)}{1 + \tan(a)\tan(b)}$

بالشكل $\tan(a) - \tan(b) = (1 + \tan(a)\tan(b))\tan(a-b)$ أفضل من استعمال تغيير المتغير من البداية.

$$\begin{aligned} \lim_{x \rightarrow \frac{p}{6}} \left(\frac{\cos x - \sqrt{3} \sin x}{6x - p} \right) &= \lim_{x \rightarrow \frac{p}{6}} \left(\frac{2 \left(\frac{1}{2} \cos x - \frac{\sqrt{3}}{2} \sin x \right)}{6 \left(x - \frac{p}{6} \right)} \right) = \lim_{x \rightarrow \frac{p}{6}} \left(\frac{1}{3} \frac{\left(\sin \left(\frac{p}{6} \right) \cos x - \cos \left(\frac{p}{6} \right) \sin x \right)}{\left(x - \frac{p}{6} \right)} \right) \\ \lim_{x \rightarrow \frac{p}{6}} \left(\frac{\cos x - \sqrt{3} \sin x}{6x - p} \right) &= \lim_{x \rightarrow \frac{p}{6}} \left(\frac{1}{3} \frac{\sin \left(\frac{p}{6} - x \right)}{\left(x - \frac{p}{6} \right)} \right) = \lim_{x \rightarrow \frac{p}{6}} \left(\frac{-1}{3} \frac{\sin \left(x - \frac{p}{6} \right)}{\left(x - \frac{p}{6} \right)} \right) = \lim_{t \rightarrow 0} \left(\frac{-1}{3} \frac{\sin(t)}{t} \right) = \frac{-1}{3} \end{aligned}$$

👉 قمنا بتغيير المتغير x وذلك بوضع $t = x - \frac{p}{6}$ ، كما يمكن إجراء تغيير المتغير منذ البداية.

تمرين 4 : $f(x) = x^2 \sin \left(\frac{1}{x^2} \right)$

لدينا : $\forall x \in \mathbb{R}^* \quad -1 \leq \sin \left(\frac{1}{x^2} \right) \leq 1$ منه : $\forall x \in \mathbb{R}^* \quad -x^2 \leq f(x) \leq x^2$

بما أن : $\lim_{x \rightarrow 0} x^2 = 0$ و $\lim_{x \rightarrow 0} -x^2 = 0$ فإن : $\lim_{x \rightarrow 0} f(x) = 0$

نضع : $t = \frac{1}{x^2}$ إذن : $\lim_{x \rightarrow +\infty} f(x) = \lim_{x \rightarrow +\infty} x^2 \sin \left(\frac{1}{x^2} \right) = \lim_{t \rightarrow 0} \frac{1}{t} \sin(t) = 1$