تصحيح الفرض المحروس رقم 2

التمرين 1

$f(x) = x+2-\sqrt{x^2-2x+4}$

$D_f = \mathbb{R}$ — ($\forall x\in\mathbb{R}\quad x^2-2x+4>0$ لأن)

$\lim\limits_{x\to+\infty} f(x) = \lim \dfrac{(x+2)^2-\sqrt{x^2-2x+4}^2}{(x+2)+\sqrt{x^2-2x+4}}$

$= \lim \dfrac{6x}{\left(1+\frac{2}{x}+\sqrt{1-\frac{2}{x}+\frac{4}{x^2}}\right)x}$

$= \dfrac{6}{2} = 3$

$y=3$ مقارب أفقي لـ (C_f) عند $+\infty$ ⇐

$\lim\limits_{x\to-\infty} f(x) = -\infty$ —

$\lim\limits_{x\to-\infty} x+2 = -\infty$

$\lim\limits_{x\to-\infty} \sqrt{x^2-2x+4} = +\infty$

$\lim\limits_{x\to-\infty} \dfrac{f(x)}{x} = \lim\limits_{-\infty} \dfrac{x+2-\sqrt{x^2-2x+4}}{x}$

$= \lim\limits_{-\infty} 1+\frac{2}{x}+\sqrt{1-\frac{2}{x}+\frac{4}{x^2}}$

$= 2$

$\lim\limits_{x\to-\infty} f(x)-2x = \lim\limits_{-\infty} (2-x)-\sqrt{x^2-2x+4}$

$= \lim\limits_{x\to-\infty} \dfrac{-2x}{\frac{2}{x}-1-\sqrt{1-\frac{2}{x}+\frac{4}{x^2}}}$

$= 1$

$y=2x+1$ مقارب مائل لـ (C_f) عند $-\infty$ ⇐

$f'(x) = (x+2)' - \dfrac{(x^2-2x+4)'}{2\sqrt{x^2-2x+4}}$ —

$f'(x) = 1-\dfrac{2x-2}{2\sqrt{x^2-2x+4}}$

$= 1-\dfrac{x-1}{\sqrt{x^2-2x+4}}$

$(x-1)^2 < x^2-2x+4$

$x-1 < |x-1| < \sqrt{x^2-2x+4}$

$f'(x)>0$ ومنه f تزايدية قطعا على $\mathbb{R}$

x	$-\infty$	$+\infty$
$f(x)$	$-\infty$ ↗	3

$f(x)-x = \dfrac{-x^2+2x}{2+\sqrt{x^2-2x+4}}$

$\forall x\in\mathbb{R}-]0;2[\Leftarrow -x^2+2x \le 0$ ⇐ C_f يوجد تحت (D)

$\forall x\in]0;2[\quad -x^2+2x>0$ ⇐ C_f يوجد فوق (D)

$U_{n+1} = f(U_n) \qquad U_0 = 1$

$U_0 = 1$

ـ نبين أن $0<U_n<2$ ونبين أن $0<U_{n+1}<2$

لدينا $0<U_n<2$ و f تزايدية

$f(0)<f(U_n)<f(2)$

$0<U_{n+1}<2$

$0<U_n<2$

ـ لدينا $f(x)>x \quad \forall x\in]0;2[$

$f(U_n)>U_n \qquad U_n\in]0;2[$

$U_{n+1}>U_n$

$(U_n)_n$ تزايدية

ـ $(U_n)_n$ تزايدية

$(U_n)_n$ مصغورة ب $U_0=1$

$\forall n\in\mathbb{N} \quad 1\le U_n<2$

$|U_{n+1}-2| = |U_n-\sqrt{U_n^2-2U_n+4}|$

$= \dfrac{2|U_n-2|}{U_n+\sqrt{U_n^2-2U_n+4}}$

$\sqrt{U_n^2-2U_n+4} \ge \sqrt{3} \ge \frac{3}{2} \qquad U_n\ge 1$

$\dfrac{1}{U_n+\sqrt{U_n^2-2U_n+4}} < \dfrac{2}{5}$

$|U_{n+1}-2| \le \frac{4}{5}|U_n-2|$

ـ من أجل $n=0$

$|U_n-2| = |-1| = 1 \le 1$

نفترض $|U_n-2| \le \left(\frac{4}{5}\right)^n$

ونبين $|U_{n+1}-2| \le \left(\frac{4}{5}\right)^{n+1}$

$|U_{n+1}-2| \le \frac{4}{5}|U_n-2|$

وبما أن $|U_n-2| \le \left(\frac{4}{5}\right)^n$

فإن $|U_{n+1}-2| \le \left(\frac{4}{5}\right)^{n+1}$

$\forall n\in\mathbb{N} \quad |U_n-2| \le \left(\frac{4}{5}\right)^n$

ـ لدينا $|U_k-2| \le \left(\frac{4}{5}\right)^k \qquad U_k<2$

$2-\left(\frac{4}{5}\right)^k \le U_k < 2 \Leftarrow 2-U_k \le \left(\frac{4}{5}\right)^k$

$\sum\limits_0^{n-1} 2-\sum\limits_0^{n-1}\left(\frac{4}{5}\right)^k \le \sum U_k \le \sum\limits_0^{n-1} 2$

$2n-5\left(1-\frac{4^n}{5^n}\right) \le \sum U_k \le 2n$

⇐ $2-\frac{5}{n}-\frac{4^n}{5^{n-1}} \le S_n \le 2$

التمرين 2

$$T_n = \sum_{k=0}^{n} \frac{(-1)^k}{2k+1}$$

$u_0 = 1$

$v_0 = \frac{2}{3}$

$$u_n - v_n = T_{2n} - T_{2n+1}$$
$$= \sum_{0}^{2n} \frac{(-1)^k}{2k+1} - \sum_{0}^{2n+1} \frac{(-1)^k}{2k+1}$$
$$= \frac{(-1)^{2n+1}}{4n+3} > 0 \Leftarrow$$

$\forall n \in \mathbb{N} \quad v_n < u_n$

$$v_{n+1} - v_n = T_{2n+3} - T_{2n+1}$$
$$= \sum_{0}^{2n+3} \frac{(-1)^k}{2k+1} - \sum_{0}^{2n+1} \frac{(-1)^k}{2k+1}$$
$$= \frac{1}{4n+5} - \frac{1}{4n+7} > 0$$

تزايدية $\Leftarrow$

$$u_{n+1} - u_n = T_{2n+1} - T_{2n}$$
$$= \frac{(-1)^{2n+1}}{4n+3} + \frac{(-1)^{2n+2}}{4n+5} < 0$$

تناقصية $\Leftarrow$

$T_n = T_{2p} = u_p \quad n = 2p$

$v_n < u_n$ تناقصية (u_n)

$$v_0 = \frac{2}{3} \leq T_n = u_p \leq 1 = u_0$$

$n = 2p+1$

$T_n = v_p$

$$\frac{2}{3} = v_0 \leq v_p < u_p \leq u_0 = 1$$

$$\frac{2}{3} \leq T_n \leq 1$$

محدودة (T_n)

التمرين 3

$1 = x \cdot y \Rightarrow x = 1$ و $y = 1$

$h(1) = 2h(1)$

$h(1) = 0$

$$1 = h'(1) = \lim_{x \to 1} \frac{h(x)}{x-1}$$

ليكن a من $]0;+\infty[$ وندرس قابلية اشتقاق h في a

$$\lim_{x \to a} \frac{h(x) - a}{x - a} = \lim_{x \to a} \frac{h(x) - h(a)}{a\left(\frac{x}{a} - 1\right)}$$

$X \to 1 \quad x \to a \quad X = \frac{x}{a}$

$$\lim_{X \to 1} \frac{1}{a} \frac{h(Xa) - h(a)}{X - 1}$$

$$\lim_{X \to a} \frac{1}{a} \left(\frac{h(X)}{X-1}\right) = \frac{1}{a}$$

h قابلة للاشتقاق على $]0;+\infty[$

$h'(x) = \frac{1}{x}$

التمرين 4

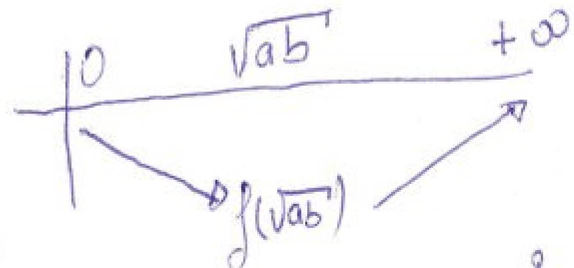
نضع $\forall x \in]0;+\infty[$

$$f(x) = a^3 + b^3 + x^3 - 3abx$$

لدينا

$$f'(x) = 3x^2 - 3ab = 3(x^2 - ab)$$

أي



$$f(\sqrt{ab}) = (a\sqrt{a} - b\sqrt{b})^2 \geq 0$$

أي $\forall x \in]0;+\infty[: f(x) \geq 0$

لدينا $c \in]0;+\infty[$

$$f(c) = a^3 + b^3 + c^3 - 3abc \geq 0$$

ومنه

$$a^3 + b^3 + c^3 \geq 3abc$$

saad ELAZZAB